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இலங்கைப் பரீட்சைத் திணைக்களம்
Department of Examinations, Sri Lanka

අධ්‍යයන පොදු සහතික පත්‍ර (උසස් පෙළ) විභාගය, 2017 අගෝස්තු
கல்விப் பொதுத் தராதரப் பத்திர (உயர் தர)ப் பரீட்சை, 2017 ஓகஸ்ற்
General Certificate of Education (Adv. Level) Examination, August 2017

සංයුක්ත ගණිතය I
இணைந்த கணிதம் I
Combined Mathematics I

10 E I

Part B

※ Answer **five** questions only.

11. (*a*) Let $f(x) = 3x^2 + 2ax + b$, where $a, b \in \mathbb{R}$.

It is given that the equation $f(x) = 0$ has two real distinct roots. Show that $a^2 > 3b$.

Let α and β be the roots of $f(x)=0$. Write down $\alpha+\beta$ in terms of a and $\alpha\beta$ in terms of b.

Show that $|\alpha - \beta| = \frac{2}{3}\sqrt{a^2 - 3b}$.

Show further that the quadratic equation with $|\alpha + \beta|$ and $|\alpha - \beta|$ as its roots is given by $9x^2 - 6\left(|a| + \sqrt{a^2 - 3b}\right)x + 4\sqrt{a^4 - 3a^2 b} = 0$.

(*b*) Let $g(x) = x^3 + px^2 + qx + 1$, where $p, q \in \mathbb{R}$. When $g(x)$ is divided by $(x-1)(x+2)$, the remainder is $3x+2$. Show that the remainder when $g(x)$ is divided by $(x-1)$ is 5, and that the remainder when $g(x)$ is divided by $(x+2)$ is -4.

Find the values of p and q, and show that $(x+1)$ is a factor of $g(x)$.

12. (*a*) Write down the binomial expansion of $(5 + 2x)^{14}$ in ascending powers of x.

Let T_r be the term containing x^r in the above expansion for $r = 0, 1, 2, \ldots, 14$.

Show that $\frac{T_{r+1}}{T_r} = \frac{2(14-r)}{5(r+1)}x$ for $x \neq 0$.

Hence, find the value of r which gives the largest term of the above expansion, when $x = \frac{4}{3}$.

(*b*) Let $c \geq 0$. Show that $\frac{2}{(r+c)(r+c+2)} = \frac{1}{(r+c)} - \frac{1}{(r+c+2)}$ for $r \in \mathbb{Z}^+$.

Hence, show that $\sum_{r=1}^{n} \frac{2}{(r+c)(r+c+2)} = \frac{(3+2c)}{(1+c)(2+c)} - \frac{1}{(n+c+1)} - \frac{1}{(n+c+2)}$ for $n \in \mathbb{Z}^+$.

Deduce that the infinite series $\sum_{r=1}^{\infty} \frac{2}{(r+c)(r+c+2)}$ converges and find its sum.

By using this sum with suitable values for c, show that $\sum_{r=1}^{\infty} \frac{1}{r(r+2)} = \frac{1}{3} + \sum_{r=1}^{\infty} \frac{1}{(r+1)(r+3)}$.

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13.(*a*) Let $\mathbf{A} = \begin{pmatrix} 2 & a & 3 \\ -1 & b & 2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & -1 & a \\ 1 & b & 0 \end{pmatrix}$ and $\mathbf{P} = \begin{pmatrix} 4 & 1 \\ 2 & 0 \end{pmatrix}$, where $a, b \in \mathbb{R}$.

It is given that $\mathbf{AB}^{\mathrm{T}} = \mathbf{P}$, where $\mathbf{B}^{\mathrm{T}}$ denotes the transpose of the matrix $\mathbf{B}$. Show that $a = 1$ and $b = -1$, and with these values for a and b, find $\mathbf{B}^{\mathrm{T}}\mathbf{A}$.

Write down $\mathbf{P}^{-1}$, and using it, find the matrix $\mathbf{Q}$ such that $\mathbf{PQ} = \mathbf{P}^2 + 2\mathbf{I}$, where $\mathbf{I}$ is the identity matrix of order 2.

(*b*) Sketch in an Argand diagram, the locus C of the points representing complex numbers z satisfying $|z| = 1$.

Let $z_0 = a(\cos\theta + i\sin\theta)$, where $a > 0$ and $0 < \theta < \frac{\pi}{2}$. Find the modulus in terms of a and the principal argument, in terms of θ, of each of the complex numbers $\frac{1}{z_0}$ and z_0^2.

Let P, Q, R and S be the points in the above Argand diagram representing the complex numbers z_0, $\frac{1}{z_0}$, $z_0 + \frac{1}{z_0}$ and z_0^2, respectively.

Show that when the point P lies on C above,

(i) the points Q and S also lie on C, and

(ii) the point R lies on the real axis between 0 and 2.

14. (*a*) Let $f(x) = \dfrac{x^2}{(x-1)(x-2)}$ for $x \neq 1, 2$.

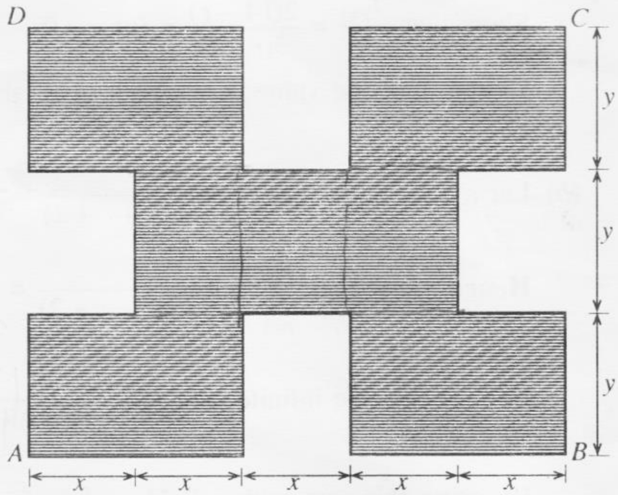
Show that $f'(x)$, the derivative of $f(x)$, is given by $f'(x) = \dfrac{x(4-3x)}{(x-1)^2(x-2)^2}$ for $x \neq 1, 2$.

Sketch the graph of $y = f(x)$ indicating the asymptotes and the turning points.

Using the graph, solve the inequality $\dfrac{x^2}{(x-1)(x-2)} \leq 0$.

(*b*) The shaded region shown in the adjoining figure is of area 385 m². This region is obtained by removing four identical rectangles each of length y metres and width x metres from a rectangle $ABCD$ of length $5x$ metres and width $3y$ metres. Show that $y = \dfrac{35}{x}$ and that the perimeter P of the shaded region, measured in metres, is given by $P = 14x + \dfrac{350}{x}$ for $x > 0$.

Find the value of x such that P is minimum.



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15.(*a*) (i) Express $\dfrac{1}{x(x+1)^2}$ in partial fractions and **hence**, find $\displaystyle\int \frac{1}{x(x+1)^2}\,dx$.

(ii) Using integration by parts, find $\displaystyle\int xe^{-x}\,dx$ and **hence**, find the area of the region enclosed by the curve $y = xe^{-x}$ and the straight lines $x = 1$, $x = 2$ and $y = 0$.

(*b*) Let $c > 0$ and $I = \displaystyle\int_0^c \frac{\ln(c+x)}{c^2+x^2}\,dx$. Using the substitution $x = c\tan\theta$,

show that $I = \dfrac{\pi}{4c}\ln c + \dfrac{1}{c}J$, where $J = \displaystyle\int_0^{\frac{\pi}{4}} \ln(1+\tan\theta)\,d\theta$.

Using the formula $\displaystyle\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$, where a is a constant, show that $J = \dfrac{\pi}{8}\ln 2$.

Deduce that $I = \dfrac{\pi}{8c}\ln(2c^2)$.

16. Let $m \in \mathbb{R}$. Show that the point $P \equiv (0, 1)$ does not lie on the straight line l given by $y = mx$.

Show that the coordinates of any point on the straight line through P perpendicular to l can be written in the form $(-mt,\ t+1)$, where t is a parameter.

Hence, show that the coordinates of the point Q, the foot of the perpendicular drawn from P to l, are given by $\left(\dfrac{m}{1+m^2}, \dfrac{m^2}{1+m^2}\right)$.

Show that, as m varies, the point Q lies on the circle S given by $x^2 + y^2 - y = 0$, and sketch the locus of Q in the xy-plane.

Also, show that the point $R \equiv \left(\dfrac{\sqrt{3}}{4}, \dfrac{1}{4}\right)$ lies on S.

Find the equation of the circle S' whose centre lies on the x-axis, and touches S externally at the point R.

Write down the equation of the circle having the same centre as that of S' and touching S internally.

17. (*a*) (i) Show that $\dfrac{2\cos(60^\circ - \theta) - \cos\theta}{\sin\theta} = \sqrt{3}$ for $0^\circ < \theta < 90^\circ$.

(ii) In the quadrilateral $ABCD$ shown in the figure, $AB = AD$, $A\hat{B}C = 80^\circ$, $C\hat{A}D = 20^\circ$ and $B\hat{A}C = 60^\circ$.

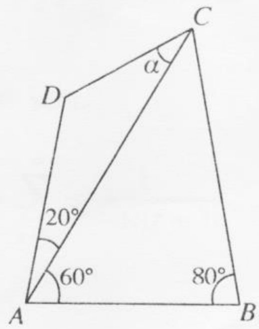
Let $A\hat{C}D = \alpha$. Using the Sine Rule for the triangle ABC, show that $\dfrac{AC}{AB} = 2\cos 40^\circ$.

Next, using the Sine Rule for triangle ADC, show that $\dfrac{AC}{AD} = \dfrac{\sin(20^\circ + \alpha)}{\sin\alpha}$.

Deduce that $\sin(20^\circ + \alpha) = 2\cos 40^\circ \sin\alpha$.

Hence, show that $\cot\alpha = \dfrac{2\cos 40^\circ - \cos 20^\circ}{\sin 20^\circ}$.

Now, using the result in (i) above, show that $\alpha = 30^\circ$.



(*b*) Solve the equation $\cos 4x + \sin 4x = \cos 2x + \sin 2x$.

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